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SOME RESULTS ON THE EMPIRICAL SPACINGS PROCESS AND ITS BOOTSTRAPPED VERSION

S. RAO JAMMALAMADAKA[1] and JING QIN[2]
[1]University of California, Santa Barbara

[2]University of Maryland

ABSTRACT

This paper considers the CLT and SLLN for the empirical spacings process, indexed by functions. A bootstrapped version of this process is shown to work and a strong approximation rate established for this bootstrapped version.

1. INTRODUCTION AND PRELIMINARIES

Let $X_1, X_2, \ldots, X_{n-1}$ be independently and identically distributed (i.i.d.) random variables (r.v.s) with a common continuous distribution function (d.f.) F with support on R^1. An important basic question of interest is whether these observations are from a specified distribution function — the goodness of fit problem. A simple probability integral transformation on these random variables, lets us equate the specified distribution to the uniform distribution on $[0,1]$. Thus from now on, we shall assume that this reduction has been done and under the null hypothesis of interest, the observations have a $U(0,1)$ distribution. A broad class of procedures for testing this null hypothesis are based on the spacings, namely

$$D_{i:n} = (X_{i,n} - X_{i-1,n}), \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

where $0 = X_{0,n} \leqslant X_{1,n} \leqslant \cdots \leqslant X_{n-1,n} \leqslant X_{n,n} = 1$ are the order statistics from $U(0,1)$ distribution. See for instance (Pyke, 1965, 1972; Rao and Sethuraman, 1975; Shorack, 1972; Aly *et al.*, 1984).

If Z_i, $i = 1, 2, \ldots$, is a sequence of i.i.d. exponential random variables with mean 1, i.e., with d.f.

$$F(z) = 1 - e^{-z}, \tag{1.2}$$

then it is well known (see, e.g. (Pyke, 1965)) that

$$\{nD_{i:n}\}_{i=1}^{n} \sim \left\{\frac{Z_i}{\overline{Z}_n}\right\}_{i=1}^{n}, \tag{1.3}$$

where $\overline{Z}_n = \sum_{i=1}^n Z_i/n$ and $\sim$ denotes equivalence in distribution. Indeed there is a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ on which both $D_{i:n}$ and Z_i can be defined, so that $\sim$ in (1.3) can be replaced by $=$ almost surely. Let

$$F_n(t) = n^{-1} \sum_{i=1}^{n} I(Z_i \leqslant t) \tag{1.4}$$

and

$$\widehat{F}_n(t) = n^{-1} \sum_{i=1}^{n} I\left(\frac{Z_i}{\overline{\overline{Z}}} \leqslant t\right) = F_n(\overline{Z}t), \tag{1.5}$$

denote the empirical d.f. (e.d.f.) of the exponential sequence and the normalized spacings sequence $\{nD_{i:n}\}_{i=1}^n$ respectively. We call

$$\hat{\alpha}_n(t) = \sqrt{n}\big(\widehat{F}_n(t) - F(t)\big), \qquad t \geqslant 0, \tag{1.6}$$

the empirical spacings process. Denote by

$$\nu_n(\cdot) = \sqrt{n}\big(P_n(\cdot) - P(\cdot)\big) \tag{1.7}$$

and

$$\hat{\nu}_n(\cdot) = \sqrt{n}\big(\widehat{P}_n(\cdot) - P(\cdot)\big), \tag{1.8}$$

where $\widehat{P}_n, P_n$, and P are probability measures corresponding to $\widehat{F}_n, F_n$, and F respectively.

Limit theory for empirical processes has grown enormously over the last few decades, as evidenced by the voluminous book by Shorack and Wellner (1986). Recently the study of empirical processes indexed by sets or functions has become very important and the results are quite general in scope. See, for instance (Pollard, 1984; Sheehy and Wellner, 1992), etc. In Section 2, we explore the central limit theorem (CLT) and the strong law of large numbers (SLLN) for the spacings process, indexed by functions. It is easily checked that the usual bootstrap method fails for the spacings process, so in Section 3, we introduce a resampling scheme for which the bootstrap approximation is shown to work. This allows one to get critical values for any spacings test statistics, which are known to be notoriously slow in converging to their limiting distributions. Finally in Section 4, a bootstrap strong approximation rate is derived.

2. SPACINGS PROCESSES INDEXED BY VC FUNCTIONS

For $1 \leqslant s < \infty$ and for some probability measure Q on R^d, we denote by $L^s(R^d, Q)$ the space of measurable real functions g on R^d with $(\int |g|^s \, \mathrm{d}Q)^{1/s} < \infty$. In most of what follows, Q will be empirical measure P_n or the theoretical measure P. We denote the $L^s(R^d, P_n)$ (pseudo) norm by

$$\|\cdot\|_{s,n} = \left(\int |\cdot|^s \, \mathrm{d}P_n\right)^{1/s}$$

and we sometimes call this the "empirical norm" whose theoretical counterpart is

$$\|\cdot\|_s = \left(\int |\cdot|^s \, \mathrm{d}P\right)^{1/s}.$$

For $\mathcal{F}$ a class of functions, the envelope H of $\mathcal{F}$ is defined as

$$H = \sup_{f \in \mathcal{F}} |f|.$$

Moreover, for $\mathcal{F} \subseteq L^s(R^d, Q)$, we define the covering number $N_s(\delta, Q, \mathcal{F})$ as the smallest value of k such that there exists $f_1, \ldots, f_k$ in $\mathcal{F}$, for all $f \in \mathcal{F}$

$$\min_{1 \leqslant j \leqslant k} \left(\int |f - f_j|^s \, \mathrm{d}Q\right)^{1/s} < \delta. \tag{2.1}$$

The logarithm of $N_s(\delta, Q, \mathcal{F})$ is called the δ-entropy of $\mathcal{F}$ for the metric

$$\left(\int |\cdot|^s \, \mathrm{d}Q\right)^{1/s}.$$

LEMMA 1. *Suppose $\mathcal{F}$ is a permissible (measurable) class with envelope H, then*

$$\sup_{f \in \mathcal{F}} \left| \int f \, \mathrm{d}(P_n - P) \right| \longrightarrow 0, \tag{2.2}$$

almost surely iff both $H \in L^1(R^d, P)$ and

$$n^{-1} \log N_1(\delta, P_n, \mathcal{F}) \longrightarrow 0, \qquad P. \tag{2.3}$$

Moreover if $H \in L^1(R^d, P)$ and (2.3) *holds, then for all $\delta > 0$, the theoretical covering number $N_1(\delta, P, \mathcal{F})$ is finite*, i.e.,

$$N_1(\delta, P, \mathcal{F}) < \infty. \tag{2.4}$$

Proof. See (for example (Van de Geer, 1988, Theorem 2.2.1 and 2.2.2)).

THEOREM 1 (Strong law of large numbers). *For a given class of functions $\mathcal{F}$, define*

$$\mathcal{F}_\alpha = \left\{ f(ct) \mid f(t) \in \mathcal{F}, \ c \in [1-\alpha, 1+\alpha] \right\},$$

where $0 < \alpha < 1$. Let H be the envelope for $\mathcal{F}_\alpha$. If

(i) $$n^{-1}\log N_1(\delta, P_n, \mathcal{F}_\alpha) \longrightarrow 0, \quad P, \tag{2.5}$$

(ii) $$\int_0^\infty H(y)\exp\left(\frac{\alpha-1}{1+\alpha}y\right)\,\mathrm{d}y < \infty \tag{2.6}$$

and

(iii) $g_f(c) = E\{f(cZ_i)\}$ *are continuous for any* $f \in \mathcal{F}$,

then

$$\sup_{f\in\mathcal{F}}\left|n^{-1}\sum_{i=1}^{n} f\left(\frac{Z_i}{\overline{\overline{Z}}}\right) - Pf\right| \longrightarrow 0 \qquad \text{a.s.} \tag{2.7}$$

Proof. When $|1/\overline{\overline{Z}} - 1| < \delta \leqslant \alpha$, we have

$$\begin{aligned}
&\sup_{f\in\mathcal{F}}\left|n^{-1}\sum_{i=1}^{n} f\left(\frac{Z_i}{\overline{\overline{Z}}}\right) - Pf\right| \\
&\quad\leqslant \sup_{f\in\mathcal{F}}\left|n^{-1}\sum_{i=1}^{n} f\left(\frac{Z_i}{\overline{\overline{Z}}}\right) - g_f\left(\frac{1}{\overline{\overline{Z}}}\right)\right| + \sup_{f\in\mathcal{F}}\left|g_f\left(\frac{1}{\overline{\overline{Z}}}\right) - g_f(1)\right| \\
&\quad\leqslant \sup_{f\in\mathcal{F},\, c\in[1-\delta,1+\delta]}\left|n^{-1}\sum_{i=1}^{n} f(cZ_i) - g_f(c)\right| \\
&\qquad + \sup_{f\in\mathcal{F}}\left|g_f\left(\frac{1}{\overline{\overline{Z}}}\right) - g_f(1)\right| \\
&\quad\leqslant \sup_{f\in\mathcal{F}_\alpha}\left|n^{-1}\sum_{i=1}^{n} f(Z_i) - Pf\right| + \sup_{f\in\mathcal{F}}\left|g_f\left(\frac{1}{\overline{\overline{Z}}}\right) - g_f(1)\right|.
\end{aligned}$$

Since $\mathcal{F}_\alpha$ satisfies (2.3), so does $\mathcal{F}$. By Lemma 1, (2.4) is true. For any $\delta > 0$, there exist $f_1, f_2, \ldots, f_k \in \mathcal{F}$ such that $k < \infty$ and (2.1) holds true with $Q = P$ and $s = 1$. Note that

$$\begin{aligned}
|g_f(c) - g_f(1)| &= |P[f(cZ) - f(Z)]| \\
&\leqslant |P[f_j(cZ) - f_j(Z)]| + |P[f(cZ) - f_j(cZ)]| + |P[f(Z) - f_j(Z)]|,
\end{aligned}$$

while

$$\begin{aligned}
|P[f(cZ) - f_j(cZ)]| &= \left|\int_0^\infty [f(cz) - f_j(cz)]\,\exp(-z)\,\mathrm{d}z\right| \\
&\leqslant c^{-1}\int |f(y) - f_j(y)|\,\exp(-y/c)\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (1-\alpha)^{-1}\left\{\int |f(y)-f_j(y)|\,\exp(-y)\,\mathrm{d}y\right\}^{1/2} \\
&\quad\times\left\{\int |f(y)-f_j(y)|\,\exp\,[2y(1/2-1/(1+\alpha))]\,\mathrm{d}y\right\}^{1/2} \\
&\leqslant \frac{2}{1-\alpha}\left\{\int H(y)\exp[y(\alpha-1)/(\alpha+1)]\,\mathrm{d}y\right\}^{1/2} \\
&\quad\times\left\{\int |f(y)-f_j(y)|\,\exp(-y)\,\mathrm{d}y\right\}^{1/2} \\
&\leqslant A^{1/2}\delta^{1/2},
\end{aligned}
$$

where

$$A=\left[\frac{2}{1-\alpha}\right]^2\int H(y)\exp[y(\alpha-1)/(\alpha+1)]\,\mathrm{d}y<\infty.$$

So for any $\varepsilon>0$, choose $\delta>0$ such that

$$\big|P\big[f(cZ)-f_j(cZ)\big]\big|+\big|P\big[f(Z)-f_j(Z)\big]\big|\leqslant A^{1/2}\delta^{1/2}+\delta<\varepsilon/2.$$

Hence combining the above, when $|1/\overline{Z}-1|<\delta$,

$$
\begin{aligned}
&\sup_{f\in\mathcal{F}}\left|n^{-1}\sum_{i=1}^{n} f\left(\frac{Z_i}{\overline{Z}}\right)-Pf\right| \\
&\quad\leqslant \sup_{f\in\mathcal{F}_\alpha}\left|n^{-1}\sum_{i=1}^{n} f(Z_i)-Pf\right|+k\max_{1\leqslant j\leqslant k}\left|g_{f_j}\left(\frac{1}{\overline{Z}}\right)-g_{f_j}(1)\right|+\frac{\varepsilon}{2}.
\end{aligned}
$$

Since k is finite, $g_{f_j}(\cdot)$ is continuous and $|1/\overline{Z}-1|\to 0$ a.s., and note the fact that

$$
\begin{aligned}
&P\left\{\sup_{f\in\mathcal{F}}\left|n^{-1}\sum_{i=1}^{n} f\left(\frac{Z_i}{\overline{Z}}\right)-Pf\right| I\left(\left|\frac{1}{\overline{Z}}-1\right|>\delta\right)>\varepsilon\right\} \\
&\quad\leqslant P\left\{\left|\frac{1}{\overline{Z}}-1\right|>\delta\right\}\leqslant\frac{c}{n^2}.
\end{aligned}
$$

By the Borel–Cantelli lemma, the theorem now follows.

COROLLARY 1. *Let* $\mathcal{F} = \{I(-\infty, x]\colon x \in R\}$, *then* $\mathcal{F}_\alpha = \mathcal{F}$ *and*

$$\sup_x \left| n^{-1} \sum_{i=1}^n I\left(\frac{Z_i}{\overline{\overline{Z}}} < x\right) - F(x) \right| \longrightarrow 0, \qquad \text{a.s.}$$

In order to study the central limit theorem, we need Pollard's sparseness condition. Let

$$J(\delta) = J(\delta, P, \mathcal{F}) = \int_0^\delta \left[2\log(N_2(x, P, \mathcal{F})^2/x)\right]^{1/2} \mathrm{d}x.$$

LEMMA 2 (Pollard, 1982, 1984). *Suppose that* $\mathcal{F} \subseteq L^2(R, P)$ *is permissible. Let the random covering number satisfies the uniformity condition: for each* $\eta > 0$ *and* $\varepsilon > 0$ *there exists a* $\gamma > 0$ *such that*

$$\limsup P\{J(\gamma, P_n, \mathcal{F}) > \eta\} < \varepsilon. \tag{2.8}$$

Then the central limit theorem holds for the class $\mathcal{F}$.

THEOREM 2. *Let* $\mathcal{F} \subseteq L^2(R, P)$ *be a continuous permissible function class such that* $\mathcal{F}_\alpha$ *satisfies* (2.8). *If*

$$\int H^2(y) \exp\left[-y(1-\alpha)/(1+\alpha)\right] \mathrm{d}y < \infty,$$

where H *is the envelope of* $\mathcal{F}_\alpha$, *then we have the representation*

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{i=1}^n \left[f\left(\frac{Z_i}{\overline{\overline{Z}}}\right) - Pf(Z)\right] \\ &\quad = \frac{1}{\sqrt{n}} \sum_{i=1}^n [f(Z_i) - Pf(Z)] + \sqrt{n}\left[g_f\left(\frac{1}{\overline{\overline{Z}}}\right) - g_f(1)\right] + \mathrm{o}_p(1), \end{aligned} \tag{2.9}$$

where $\mathrm{o}_p(1)$ *holds uniformly for* $f \in \mathcal{F}$.

Proof. Since $\mathcal{F}_\alpha$ satisfies (2.1), so does $\mathcal{F}$. For any $\delta > 0$, there exist $f_1, f_2, \ldots, f_k \in \mathcal{F}$, such that $k < \infty$ and (2.1) is true with $Q = P$ and $s = 2$. By Minkowski's inequality

$$\begin{aligned} &\sup_{f\in\mathcal{F}} \left\{\int [f(cZ) - f(Z)]^2 \mathrm{d}P\right\}^{1/2} \\ &\quad \leqslant \sup_{f\in\mathcal{F}} \left\{\int [f(cZ) - f_j(cZ)]^2 \mathrm{d}P\right\}^{1/2} \\ &\qquad + \sup_{f\in\mathcal{F}} \left\{\int [f_j(cZ) - f_j(Z)]^2 \mathrm{d}P\right\}^{1/2} \\ &\qquad + \sup_{f\in\mathcal{F}} \left\{\int [f_j(Z) - f(Z)]^2 \mathrm{d}P\right\}^{1/2}. \end{aligned} \tag{2.10}$$

Note that for $c \in [1-\alpha, 1+\alpha]$,

$$
\begin{aligned}
&\int \left[f(cZ) - f_j(cZ)\right]^2 \mathrm{d}P \\
&= \frac{1}{c} \int_0^\infty \left[f(y) - f_j(y)\right]^2 \exp(-y/c)\,\mathrm{d}y \\
&\leqslant \frac{2}{1-\alpha} \int H(y)\left|f(y) - f_j(y)\right| \exp(-y/2) \\
&\quad \times \exp\left[-y\left(\frac{1}{1+\alpha} - \frac{1}{2}\right)\right] \mathrm{d}y \qquad (2.11)\\
&\leqslant \frac{2}{1-\alpha} \left\{\int H^2(y) \exp\left[-y(1-\alpha)/(1+\alpha)\right] \mathrm{d}y\right\}^{1/2} \\
&\quad \times \left\{\int \left|f_j(y) - f(y)\right|^2 \exp(-y)\,\mathrm{d}y\right\}^{1/2} \\
&\leqslant (C-2)\delta,
\end{aligned}
$$

where $0 < C - 2 < \infty$. So there exists a $\delta_1(\delta) > 0$ such that if $|c-1| < \delta_1$,

$$
\sup_{f \in \mathcal{F}} \left\{\int \left[f(cZ - f(Z)\right]^2 \mathrm{d}P\right\}^{1/2} \leqslant C\delta
$$

or

$$
\begin{aligned}
[C\delta] &\equiv \left\{(f_1(t), f_2(t)) \mid f_1, f_2 \in \mathcal{F}_\alpha,\ \int (f_1 - f_2)^2 \,\mathrm{d}P < C\delta\right\} \\
&\supset \left\{(f(ct), f(t)) \mid f \in \mathcal{F},\ |c-1| < \delta_1(\delta)\right\} \\
&\equiv [\delta_1]
\end{aligned}
$$

and

$$
\begin{aligned}
&P\Big\{\sup_{f \in \mathcal{F}} \left|\nu_n(f(Z/\overline{Z})) - \nu_n(f(Z))\right| > \eta\Big\} \\
&\leqslant P\Big\{\sup_{f \in \mathcal{F}} \left|\nu_n(f(Z/\overline{Z})) - \nu_n(f(Z))\right| > \eta, |1/\overline{Z} - 1| < \delta_1\Big\} \\
&\quad + P\{|1/\overline{Z} - 1| > \delta_1\} \qquad (2.12)\\
&\leqslant P\Big\{\sup_{[\delta_1]} \left|\nu_n(f(cZ)) - \nu_n(f(Z))\right| > \eta\Big\} + P\{|1/\overline{Z} - 1| > \delta_1\} \\
&\leqslant P\Big\{\sup_{[C\delta]} \left|\nu_n(f_1(Z)) - \nu_n(f_2(Z))\right| > \eta\Big\} + P\{|1/\overline{Z} - 1| > \delta_1\}.
\end{aligned}
$$

By the stochastically equicontinuous property of $\{\nu_n\}$ (see (Pollard, 1984, p. 139–150)), for n large enough, the right hand side of (2.12) can be arbitrarily small. Hence

$$
\left|\nu_n(f(Z/\overline{Z})) - \nu_n(f(Z))\right| = o_p(1) \qquad \text{uniformly in} \quad f \in \mathcal{F},
$$

i.e.,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big[f(Z_i/\overline{Z})-g_f(1/\overline{Z})\big]-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big[f(Z_i)-Pf(Z)\big]=\mathrm{o}_p(1).$$

COROLLARY 2. *Under the conditions of Theorem* 2, *if*

$$\lim_{t\to 0}\sup_{f\in\mathcal{F}}\left|\frac{g_f(1+t)-g_f(1)}{t}-g_f'(1)\right|=0,$$

then

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big[f(Z_i/\overline{Z})-Pf(Z)\big]\longrightarrow B_f+g_f'(1)\int F^{-1}\,\mathrm{d}B_f,$$

where B_f *is a Brownian bridge indexed by* $\mathcal{F}$.

This result can be generalized to empirical processes subject to perturbations and scale factors, the more general setting described in (Rao and Sethuraman, 1975).

3. BOOTSTRAPPING FOR THE EMPIRICAL SPACINGS PROCESS

Let $X_1, X_2, \ldots$ be i.i.d. with distribution P on $(\Omega, \mathcal{A}, \mathcal{P})$ and let

$$P_n = n^{-1}\sum_{i=1}^{n} I(X_i \leqslant x)$$

be the empirical measure of the first n X's. The nonparametric bootstrap proceeds by sampling from $P_n^{\omega} = P_n(\cdot, \omega)$. Suppose that $X_{n1}^*, \ldots, X_{nm}^*$ are i.i.d. with the distribution P_n^{ω} on $(\Omega, \mathcal{A})$. Let

$$P_m^* = m^{-1}\sum_{i=1}^{m} I\big(X_{ni}^* \leqslant x\big), \qquad \nu_{n,m}^* = \sqrt{m}\big(P_m^* - P_n\big).$$

Thus $X_{n1}^*, \ldots, X_{nm}^*$ is the "bootstrap sample", P_m^* is the "bootstrap empirical measure" and $\nu_{m,n}^*$ is the "bootstrap empirical process". Under very weak conditions, Sheehy and Wellner (1992) show that bootstrap approximation for empirical processes indexed by functions works. However, for the empirical spacings processes, the usual bootstrap procedure fails, i.e., if we resample $U_1^*, \ldots, U_{m-1}^*$ from $\widetilde{F}_n(t) = n^{-1}\sum_{i=1}^{n} I(U_i \leqslant t)$, where U are uniform distribution random variables. Construct "bootstrap spacings"

$$\delta_{i:m}^* = U_{i:m}^* - U_{i-1:m}^*, \qquad i = 1, 2, \ldots, m,$$

where $0 = U^*_{0:m} \leqslant U^*_{1:m} \leqslant \cdots \leqslant U^*_{m:m} = 1$ are the order statistics of $U^*_1, \ldots, U^*_m$. Then the "bootstrap spacings process" is

$$\tilde{\nu}^*_m = \sqrt{m}\big[\widetilde{F}^*_m(t) - \widehat{F}_n(t)\big], \qquad \widetilde{F}^*_m(t) = m^{-1}\sum_{i=1}^{m} I\big(\delta^*_{i:m} \leqslant t\big).$$

In the bootstrap sample, the possibility of ties is high and this makes too many bootstrapped spacings to be zero. Thus, it can be checked that $\tilde{\nu}^*_m$ and $\hat{\nu}_n$ have different limit distributions and the usual bootstrapping fails. Hence we suggest the following modified resampling scheme:

Let $Z^*_1, \ldots, Z^*_m$ be i.i.d. random variables with the distribution function $\widehat{F}_n(t)$, the empirical spacings processes defined in (1.5) and let

$$F^*_m(t) = m^{-1}\sum_{i=1}^{m} I\big(Z^*_i \leqslant t\big),$$

$$\widehat{F}^*_m(t) = m^{-1}\sum_{i=1}^{m} I\left(\frac{Z^*_i}{\overline{Z}^*} \leqslant t\right) = F^*_m\big(\overline{Z}^* t\big).$$

We define the modified "bootstrap empirical spacing processes" as

$$\nu^*_m = \sqrt{m}\big(F^*_m(t) - \widehat{F}_n(t)\big), \qquad \hat{\nu}^*_m = \sqrt{m}\big(\widehat{F}^*_m(t) - \widehat{F}_n(t)\big). \tag{3.1}$$

We will now prove that this modified Bootstrap empirical spacings process (3.1) and the empirical spacings process have the same asymptotic distribution.

LEMMA 3 (Sheehy and Wellner, 1992). *Suppose that $\mathcal{F}$ is nearly linearly Deviation Measurable for* ($\{P_n\}$) (see (Gine and Zinn, 1984, p. 935) *for a definition*) *and that*

i) *H is $\{P_n\}$ uniformly square integrable,*

ii) *$(\mathcal{F}, H)$ is a sparse,*

iii) $\|P_n - P_0\|_{\mathcal{G}} \to 0$ *as* $n \to \infty$,

where $\mathcal{G} = \mathcal{F} \cup \mathcal{F}^2 \cup \mathcal{F}'^2$ and $\mathcal{F}^2 = \{f^2 \mid f \in \mathcal{F}\}$, $\mathcal{F}'^2 = \{(f-g)^2 \mid f \in \mathcal{F}, g \in \mathcal{F}\}$. Then $\mathcal{F} \in CLT(\{P_n\}_{n\geqslant 0})$ (i.e., central limit theorem holds uniformly with respect to P_n).

LEMMA 4. *Under the conditions of Lemma* 3, *if $\mathcal{F}$ is a continuous function class, then*

$$\big\|\widehat{P}_n - P\big\|_{\mathcal{G}_{\alpha/2}} \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty,$$

where $\mathcal{G}_{\alpha/2} = \mathcal{F}_{\alpha/2} \cup \mathcal{F}^2_{\alpha/2} \cup \mathcal{F}'^2_{\alpha/2}$.

Proof. We only need to prove

$$\|\widehat{P}_n - P\|_{\mathcal{F}'^2} = \sup_{f_1, f_2 \in \mathcal{F}_{\alpha/2}} \left|\left(\widehat{P}_n - P\right)(f_1 - f_2)^2\right| \longrightarrow 0 \qquad \text{as} \quad n \to \infty,$$

since the other conditions are the same. Let

$$g_{f,h}(c_1, c_2) = P\left[f(c_1 Z) - h(c_2 Z)\right]^2.$$

When $c_1/\overline{Z} \in [1-\alpha, 1+\alpha)$ and $c_2/\overline{Z} \in [1-\alpha, 1+\alpha)$, we have

$$\begin{aligned}
&\sup_{f_1, f_2 \in \mathcal{F}_{\alpha/2}} \left|\left(\widehat{P}_n - P\right)(f_1 - f_2)^2\right| \\
&\quad = \sup_{f,h\in\mathcal{F}} \sup_{c_1,c_2\in[1-\alpha/2,1+\alpha/2]} \left|\left(\widehat{P}_n - P\right)\left(f(c_1 Z) - h(c_2 Z)\right)^2\right| \\
&\quad \leqslant \sup_{f,h\in\mathcal{F}} \sup_{c_1,c_2\in[1-\alpha/2,1+\alpha/2]} \Bigg| n^{-1} \sum_i \left[f\left(c_1 Z_i/\overline{Z}\right) - h\left(c_2/\overline{Z}\right)\right]^2 \\
&\qquad - g_{f,h}\left(c_1/\overline{Z}, c_2/\overline{Z}\right)\Bigg| \\
&\qquad + \sup_{f,h\in\mathcal{F}} \sup_{c_1,c_2\in[1-\alpha/2,1+\alpha/2]} \left|g_{f,h}\left(c_1/\overline{Z}, c_2/\overline{Z}\right) - g_{f,h}(c_1, c_2)\right| \\
&\quad \leqslant \sup_{f,h\in\mathcal{F}_\alpha} \left| n^{-1} \sum_i \left[f(Z_i) - h(Z_i)\right]^2 - g_{f,h}(1,1)\right| \\
&\qquad + \sup_{f,h\in\mathcal{F}} \sup_{c_1,c_2\in[1-\alpha/2,1+\alpha/2]} \left|g_{f,h}\left(c_1/\overline{Z}, c_2/\overline{Z}\right) - g_{f,h}(c_1, c_2)\right|.
\end{aligned} \tag{3.2}$$

The first term of (3.2) goes to zero by the Theorem 12 in (Pollard, 1982). We need to prove that the second term of (3.2) also goes to zero. For $a \in [1-\alpha/2, 1+\alpha/2]$, consider

$$\begin{aligned}
&\left|g_{f,h}(c_1 a, c_2 a) - g_{f,h}(c_1, c_2)\right| \\
&\quad = \Big|P\left[f(c_1 a Z) - h(c_2 a Z)\right]^2 - P\left[f_i(a c_1 Z) - f_j(a c_2 Z)\right]^2 \\
&\qquad + P\left[f_i(a c_1 Z) - f_j(a c_2 Z)\right]^2 - P\left[f(c_1 Z) - h(c_2 Z)\right]^2 \\
&\qquad + P\left[f_i(c_1 Z) - f_j(c_2 Z)\right]^2 - P\left[f_i(c_1 Z) - f_j(c_2 Z)\right]^2\Big| \\
&\quad \leqslant P\left\{\left[\left|f(c_1 a Z) - f_i(c_1 a Z)\right| + \left|h(c_2 a Z) - f_j(c_2 a Z)\right|\right] 4H(Z)\right\} \\
&\qquad + P\left\{\left[\left|f(c_1 Z) - f_i(c_1 Z)\right| + \left|h(c_2 Z) - f_j(c_2 Z)\right|\right] 4H(Z)\right\} \\
&\qquad + \left|P\left[f_i(a c_1 Z) - f_j(a c_2 Z)\right]^2 - P\left[f_i(c_1 Z) - f_j(c_2 Z)\right]^2\right|.
\end{aligned}$$

Rest of the proof is similar to that of Theorem 2.2.

THEOREM 3. *Under the same conditions as in Theorem 2, for almost all sample sequences* $X_1, X_2, \ldots, X_n$

$$\begin{aligned}\sqrt{m}&\left[m^{-1}\sum_{i=1}^{m} f\left(\frac{Z_i^*}{\overline{Z}^*}\right) - n^{-1}\sum_{i=1}^{n} f\left(\frac{Z_i}{\overline{Z}}\right)\right]\\ &= \sqrt{m}\left[m^{-1}\sum_{i=1}^{m} f(Z_i^*) - n^{-1}\sum_{i=1}^{n} f(Z_i/\overline{Z})\right]\\ &\quad + \frac{\sqrt{m}}{n}\left[\sum_{i=1}^{m} f\left(\frac{Z_i}{\overline{Z}\,\overline{Z}^*}\right) - f\left(\frac{Z_i}{\overline{Z}}\right)\right] + o_p(1), \quad \text{a.s.}\end{aligned} \tag{3.3}$$

where $o_p(1)$ *is uniform for all* $f \in \mathcal{F}$.

Proof. We show that $\mathcal{F}_{\alpha/2} \in CLT(\{P_n\}_{n\geqslant 0})$ by using Lemma 3.

i) Let $H_1(t) = \sup_{f\in\mathcal{F}_{\alpha/2}} |f(t)|$ be an envelope for $\mathcal{F}_{\alpha/2}$. Since $H(t)$ is

$$\sup_{c\in(1-\alpha/2, 1+\alpha/2)} |H_1(ct)| = \sup_{c\in(1-\alpha/2,1+\alpha/2)}\ \sup_{f\in\mathcal{F}_{\alpha/2}} |f(ct)| \leqslant H(t),$$

then $H_1(t)$ is a.s. $\{\widehat{P}_n\}$ uniformly square integrable. In fact when $|1/\overline{Z} - 1| < \alpha/2$,

$$\begin{aligned}\widehat{P}_n H_1^2 I(H_1 \geqslant \lambda) &= n^{-1}\sum H_1^2(Z_i/\overline{Z}) I\big(H_1(Z_i/\overline{Z}) > \lambda\big)\\ &\leqslant n^{-1}\sum H^2(Z_i) I(H(Z_i) > \lambda) \longrightarrow P\big(H^2 I(H \geqslant \lambda)\big) \qquad \text{a.s.}\end{aligned}$$

iii) is correct for $\mathcal{G}_{\alpha/2}$ by Lemma 4. Then by Lemma 3, $\mathcal{F}_{\alpha/2} \in CLT(\{\widehat{P}_n\})$. For any $\delta > 0$, there exists a $\delta_1(\delta) > 0$, $\delta_1 < \alpha/4$ such that

$$\sup_{f\in\mathcal{F},\ |c-1|<\delta_1} P\big(f(cZ) - f(Z)\big)^2 < \delta/2. \tag{3.4}$$

Note that

$$\begin{aligned}&P^*\Big\{\sup_{f\in\mathcal{F}} |\nu_m^*(f(Z/\overline{Z}^*)) - \nu_m^*(f(Z))| > \eta\Big\}\\ &\leqslant P^*\Big\{\sup_{f\in\mathcal{F}} \big|\nu_m^*\big(f(Z/\overline{Z}^*)\big) - \nu_m^*\big(f(Z)\big)\big| > \eta,\ \big|1/\overline{Z}^* - 1\big| < \delta_1\Big\}\\ &\quad + P^*\big\{\big|1/\overline{Z}^* - 1\big| > \delta_1\big\}\\ &\leqslant P^*\Big\{\sup_{f\in\mathcal{F},\ c\in[1-\alpha/4, 1+\alpha/4]} \big|\nu_m^*\big(f(cZ)\big) - \nu_m^*\big(f(Z)\big)\big| > \eta\Big\}\\ &\quad + P^*\big\{\big|1/\overline{Z}^* - 1\big| > \delta_1\big\}.\end{aligned}$$

By Lemma 4

$$\sup_{f\in\mathcal{F},\ c\in[1-\alpha/4,1+\alpha/4]} \left|\widehat{P}_n\big(f(cZ)-f(Z)\big)\right|^2 \longrightarrow \sup_{f\in\mathcal{F},\ c\in[1-\alpha/4,1+\alpha/4]} \left|P\big(f(cZ)-f(Z)\big)\right|^2 < \delta/2,$$

i.e., the equicontinuity holds for the process. By the equivalent property between $\mathcal{F}_{\alpha/2} \in CLT(\{P_n\}_{n\geqslant 0})$ and the asymptotic equicontinuity of $\mathcal{F}_{\alpha/2}$ uniformly in P_n (see (Sheehy and Wellner, 1992)), the right hand side of the above can be made arbitrarily small for almost samples. Hence, we have

$$\nu_m^*\big(f\big(Z/\overline{Z}^*\big)\big) - \nu_m^*\big(f(Z)\big) = \mathrm{o}_p(1) \quad \text{a.s. uniformly for} \quad f \in \mathcal{F}.$$

and we have the desired results.

It is easy to see the bootstrap version of spacings processes indexed by functions has the same asymptotical distribution with the spacings processes.

4. BOOTSTRAP STRONG APPROXIMATION RATE

This section is based on the results of Aly *et al.* (1984) where they discuss the k-spacings processes. For convenience we restrict attention to bootstrapping the simple spacings process (with $k = 1$).

LEMMA 5 (Aly *et al.*, 1984). *Define a sequence of Gaussian processes* $\{\widehat{B}(t)\colon\ 0 \leqslant t < \infty\}$ *on* $(\Omega, \mathcal{A}, \mathcal{P})$ *such that*

$$\sup_{0\leqslant t<\infty} \left|\hat{\nu}_n(t) - \widehat{B}(F(t))\right| = \mathrm{O}\big(n^{-1/4}(\log n)^{3/4}\big) \qquad \text{a.s.}$$

and

$$P\Big\{\sup_{0\leqslant t<\infty} \left|\hat{\nu}_n(t)-\widehat{B}\big(F(t)\big)\right| > An^{-1/4}(\log n)^{3/4}\Big\} \leqslant Cn^{-\varepsilon} \qquad \textit{for any} \quad \varepsilon > 0,$$

where

$$\hat{\nu}_n(t) = \sqrt{n}\big(\widehat{F}_n(t) - F(t)\big), \qquad \widehat{B}(t) = B(t) + WF^{-1}(t)f\big(F^{-1}(t)\big) \tag{4.1}$$

with $W = \int_0^1 F^{-1}\,\mathrm{d}B \sim N(0,\sigma^2)$ *and* $B(t)$ *is a standard Brownian bridge.*

LEMMA 6 (Dvoretzky–Kiefer–Wolfowitz inequality).

$$P\Big(\sup_t \left|\widehat{F}_n(t) - F(t)\right| > d\Big) \leqslant c_1 \exp\big(-c_2 n d^2\big), \qquad d > 0, \tag{4.2}$$

where c_1, c_2 *are positive constants.*

Proof. We have

$$P\Big(\sup_t |\widehat{F}_n(t) - F(t)| > d\Big)$$
$$\leqslant P\Big(\sup_t \big|F_n\big(\overline{Z}t\big) - F\big(\overline{Z}t\big)\big| > d/2\Big) + P\Big(\sup_t \big|F\big(\overline{Z}t\big) - F(t)\big| > d/2\Big),$$

but $F(\overline{Z}t) - F(t) = (\overline{Z} - 1)tf(\eta t)$, where $f(\cdot)$ is the density function of exponential (1), η lies between 1 and $\overline{Z}$. Note the fact

$$\sup_{|\eta-1|<d<1,\ -\infty<t<\infty} \big|tf(\eta t)\big| = A < \infty.$$

$$P\Big(\sup_t \big|F\big(\overline{Z}t\big) - F(t)\big| > d/2\Big)$$
$$\leqslant P\big(\big|\overline{Z} - 1\big| > d/2\big) + P\Big(\sup_t \big|\overline{Z} - 1\big|\, tf(\eta t)\big| > d/2, \big|\overline{Z} - 1\big| \leqslant d/2\Big)$$
$$\leqslant P\big(\big|\overline{Z} - 1\big| > d/2\big) + P\big(A\big|\overline{Z} - 1\big| > d/2\big).$$

For exponential(1), the moment condition holds for Bernstein's inequality (see (Shorack and Wellner, 1986, p. 855), we have

$$P\big(\big|\overline{Z} - 1\big| > d/2\big) \leqslant \exp\big(- nc_1 d^2\big). \tag{4.3}$$

So by the D–K–W inequality for empirical distribution function

$$P\Big(\sup_t \big|F_n\big(\overline{Z}t\big) - F\big(\overline{Z}t\big)\big| > d/2\Big) \leqslant P\Big(\sup_t \big|F_n(t) - F(t)\big| > d/2\Big)$$
$$\leqslant c_2 \exp\big(- cnd^2\big).$$

Combining the above inequalities we have the result.

LEMMA 7. *Assume that* $0 < \lim_{n\to\infty} \inf(m/n) \leqslant \lim_{n\to\infty} \sup(m/n) < \infty$. *Then*

$$P\left\{\left|\sqrt{m}\big(\overline{Z}^* - 1\big) - \int_0^\infty t\,\mathrm{d}B\big(F(t)\big)\right| > An^{-1/2}(\log n)^2\right\} \leqslant Cn^{-\varepsilon} \tag{4.4}$$

and

$$P\big\{\big|\overline{Z}^* - 1\big| > An^{-1/2}(\log n)^{1/2}\big\} = \mathrm{O}(n^{-\varepsilon}). \tag{4.5}$$

Proof. Let $\eta_1, \eta_2, \ldots, \eta_m$ be i.i.d. $U(0,1)$, then $F_m^*(t) = U_m(\widehat{F}_n(t))$, where

$$U_m(t) = \frac{1}{m}\sum_{i=1}^{m} I(U_i < t), \qquad \alpha_m(t) = \frac{1}{\sqrt{m}}\sum_{i=1}^{m}\big[I(U_i < t) - t\big].$$

Note that

$$\overline{Z}^* - 1 = \int_0^\infty t\,\mathrm{d}\big(F_m^*(t) - \widehat{F}_n(t)\big) = -\int_0^\infty [F_m^*(t) - \widehat{F}_n(t)]\,\mathrm{d}t$$

$$= -\int_0^\infty [U_m(\widehat{F}_n(t)) - \widehat{F}_n(t)]\,\mathrm{d}t = -\frac{1}{\sqrt{m}}\int_0^\infty \alpha_m(\widehat{F}_n(t))\,\mathrm{d}t$$

and

$$\left|\sqrt{m}(\overline{Z}^* - 1) + \int_0^\infty B(F(t))\,\mathrm{d}t\right|$$

$$= \left|\int_0^\infty \big(\alpha_m(\widehat{F}_n(t)) - B(F(t)\big)\,\mathrm{d}t\right|$$

$$= \left|\int_0^{Z_{(n)}/\overline{Z}} \big(\alpha_m(\widehat{F}_n(t)) - B(F(t)\big)\,\mathrm{d}t\right| + \left|\int_{Z_{(n)}/\overline{Z}}^\infty \big(0 - B(F(t)\big)\,\mathrm{d}t\right|$$

$$\leqslant \frac{Z_{(n)}}{\overline{Z}} \sup_{0\leqslant t<\infty} \left|\alpha_m(\widehat{F}_n(t)) - B(F(t))\right| + \int_{Z_{(n)}/\overline{Z}}^\infty \left|B(F(t))\right|\mathrm{d}t.$$

Observe that

$$P\{Z_{(n)} > c\log n\} = 1 - P^n\{Z < c\log n\} = 1 - \{1 - n^{-c}\}^n = \mathrm{O}(n^{-\varepsilon})$$

when $c > 1$. (In fact $P(\lim_{n\to\infty} Z_{(n)}/\log n = 1) = 1$, see (Galambos, 1978, p. 224). Rest of the proof is similar to that in (Aly *et al.*, 1984, Lemma 3.1 and 3.2.).

LEMMA 8. *Under the same conditions of Lemma* 7,

$$P\left\{\left|\sqrt{m}\big(\widehat{F}_n(\overline{Z}^* t) - \widehat{F}_n(t)\big) - tf(t)\int_0^\infty t\,\mathrm{d}B(F(t))\right| > n^{-1/4}(\log n)^{3/4}\right\} = \mathrm{O}(n^{-\varepsilon}). \tag{4.6}$$

Proof. We have

$$\begin{aligned}
&\sqrt{m}[\widehat{F}_n(\overline{Z}^* t) - \widehat{F}_n(t)] \\
&\quad = \sqrt{m}[\widehat{F}_n(\overline{Z}^* t) - F(\overline{Z}^* t)] - \sqrt{m}[\widehat{F}_n(t) - F(t)] + \sqrt{m}[F(\overline{Z}^* t) - F(t)] \\
&\quad = \sqrt{m/n}[\hat{\nu}_n(\overline{Z}^* t) - \hat{\nu}_n(t)] + \sqrt{m}[F(\overline{Z}^* t) - F(t)] \\
&\quad = \sqrt{m/n}\{[\hat{\nu}_n(\overline{Z}^* t) - \widehat{B}(F(\overline{Z}^* t))] - [\hat{\nu}_n(t) - \widehat{B}(F(t))]\} \\
&\qquad + \sqrt{m/n}[\widehat{B}(F(\overline{Z}^* t)) - \widehat{B}(F(t))] + \sqrt{m}(F(\overline{Z}^* t) - F(t)).
\end{aligned}$$

By one-term Taylor expansion, we have

$$\left|F\left(\overline{Z}^{*}t\right)-F(t)\right| = f(t\eta)t\left|\overline{Z}^{*}-1\right|,$$

where η lies between 1 and $\overline{Z}^{*}$. Let $A_n(\delta) = \{\omega\colon\ |\overline{Z}^{*}-1| < \delta\}$, n large enough to make $An^{-1/2}(\log n)^{1/2} < \delta$, then $P\{A_n^c(\delta)\} \leqslant Bn^{-\varepsilon}$ and on $A_n(\delta)$, we have

$$tf(t\eta) \leqslant \sup_{1-\delta<\eta<1+\delta,0\leqslant t<\infty} tf(t\eta) = M < \infty.$$

Then by using Lemma 7,

$$\begin{aligned}
&P\left\{\sup_{0\leqslant t<\infty}\left|F\left(\overline{Z}^{*}t\right)-F(t)\right| > Mn^{-1/2}(\log n)^{1/2}\right\}\\
&\quad\leqslant P\{A_n^c(\delta)\}\\
&\qquad + P\left\{A_n(\delta)\bigcap\left\{\sup_{0\leqslant t<\infty}|F(\overline{Z}^{*}t)-F(t)| > Mn^{-1/2}(\log n)^{1/2}\right\}\right\}\\
&\quad\leqslant Cn^{-\varepsilon} + P\left\{\left|\overline{Z}^{*}-1\right| > An^{-1/2}(\log n)^{1/2}\right\}\\
&\quad\leqslant 2Cn^{-\varepsilon} \quad \text{for } n \text{ large enough.}
\end{aligned}$$

Note that from (4.1)

$$\widehat{B}\left(F\left(\overline{Z}^{*}t\right)\right)-\widehat{B}\left(F(t)\right) = B\left(F\left(\overline{Z}^{*}t\right)\right)-B\left(F(t)\right)+Wt\left\{\overline{Z}^{*}f\left(\overline{Z}^{*}t\right)-f(t)\right\}.$$

Just like the proof in (Aly *et al.*, 1984, Lemma 3.2) it is easy to prove

$$P\left\{\left|\widehat{B}\left(F\left(\overline{Z}^{*}t\right)\right)-\widehat{B}\left(F(t)\right)\right| > n^{-1/4}(\log n)^{3/4}\right\} \leqslant Cn^{-\varepsilon},$$

$$P\left\{\left|\left[\hat{\nu}_n\left(\overline{Z}^{*}t\right)-\widehat{B}\left(F\left(\overline{Z}^{*}t\right)\right)\right]-\left[\hat{\nu}_n(t)-\widehat{B}\left(F(t)\right)\right]\right| > n^{-1/4}(\log n)^{3/4}\right\} \leqslant Cn^{-\varepsilon}.$$

Finally,

$$\sqrt{m}\left\{F\left(\overline{Z}^{*}t\right)-F(t)\right\} = \sqrt{m}f(t)t\left(\overline{Z}^{*}-1\right)+\sqrt{m}f'(t\eta)t^2\left(\overline{Z}^{*}-1\right)^2,$$

it is easy to get the result.

THEOREM 4. *Assume that* $0 < \lim_{n\to\infty}\inf(m/n) \leqslant \lim_{n\to\infty}\sup(m/n) < \infty$. *Then we can define a sequence of Brownian motions* $\{\widehat{B}(t),\ 0 \leqslant t \leqslant 1\}$ *such that*

$$\begin{aligned}
&P\left\{\sup_{0\leqslant t<\infty}\left|\sqrt{m}\left(\widehat{F}_m^{*}(t)-\widehat{F}_n(t)\right)-\widehat{B}\left(F(t)\right)\right| > An^{-1/4}(\log n)^{3/4}\right\}\\
&\quad\leqslant Cn^{-\varepsilon},
\end{aligned} \tag{4.7}$$

for any $\varepsilon > 0$.

Proof.

$$\sqrt{m}(\widehat{F}_m^*(t) - \widehat{F}_n(t)) = \sqrt{m}(F_m^*(\overline{Z}^*t) - F_m^*(t)) + \sqrt{m}(F_m^*(t) - \widehat{F}_n(t)).$$

$$\begin{aligned}
&\sqrt{m}(\widehat{F}_m^*(t) - \widehat{F}_n(t)) \\
&\quad = \sqrt{m}\{U_m(\widehat{F}_n(\overline{Z}^*t)) - U_m(\widehat{F}_n(t))\} + \sqrt{m}\{U_m(\widehat{F}_n(t)) - \widehat{F}_n(t)\} \\
&\quad = \{\alpha_m(\widehat{F}_n(\overline{Z}^*t)) - \alpha_m(\widehat{F}_n(t))\} + \sqrt{m}\{\widehat{F}_n(\overline{Z}^*t) - \widehat{F}_n(t)\} \\
&\qquad + \alpha_m(\widehat{F}_n(t)).
\end{aligned} \tag{4.8}$$

Using the Komlós, Major, and Tusnády's construction, we can define a sequence of Brownian bridges $B(t)$ which are independent of $(Z_1, \ldots, Z_n)$ such that

$$P\Big\{\sup_{0\leqslant t\leqslant 1} |\alpha_m(t) - B(t)| > A_{11}n^{-1/2}(\log n)^{1/2}\Big\} \leqslant B_{12}n^{-\varepsilon} \tag{4.9}$$

for any $\varepsilon > 0$ (see (Csörgö *et al.*, 1986, p. 158)). By Theorem 2c of Burke *et al.* (1981) (for example (Csörgö and Révész, 1981, Lemma 1.1.1)), we have

$$P\Big\{\sup_{0<s\leqslant 1-h_n}\ \sup_{0\leqslant t\leqslant h_n} |B(s)-B(s+t)| > A_{13}n^{-1/4}(\log n)^{3/4}\Big\} \leqslant B_{13}n^{-\varepsilon} \tag{4.10}$$

with $h_n = A_{12}n^{-1/2}$. By Lemma 6, we have

$$P\Big\{\sup_t |\widehat{F}_n(t) - F(t)| > A_{12}n^{-1/2}(\log n)^{1/2}\Big\} \leqslant B_{12}n^{-\varepsilon}.$$

Note the fact that

$$\begin{aligned}
&\sup_t |\widehat{F}_n(\overline{Z}^*t) - \widehat{F}_n(t)| \\
&\quad \leqslant \sup_t |\widehat{F}_n(\overline{Z}^*t) - F(\overline{Z}^*t)| + \sup_t |\widehat{F}_n(t) - F(t)| + \sup_t |F(\overline{Z}^*t) - F(t)| \\
&\quad \leqslant 2\sup_t |\widehat{F}_n(t) - F(t)| + \sup_t |F(\overline{Z}^*t) - F(t)|.
\end{aligned}$$

Combining the above inequality with Lemma 8, we have

$$P\Big\{\sup_t |\alpha_m(\widehat{F}_n(\overline{Z}^*t)) - \alpha_m(\widehat{F}_n(t))| > An^{-1/4}(\log n)^{3/4}\Big\} \leqslant Cn^{-\varepsilon}$$

and

$$P\Big\{\sup_{0\leqslant t<\infty} |\alpha_m(\widehat{F}_n(t)) - B(F(t))| > A_1n^{-1/4}(\log n)^{3/4}\Big\} \leqslant Cn^{-\varepsilon} \tag{4.11}$$

for any $\varepsilon > 0$ and some constants. From this, the desired result follows.

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